Graph Theory

1. *(Summer 2016)* Consider the graphs

$$K_9 \qquad K_{12} \qquad C_{2016} \qquad K_{4,6} \qquad K_{3,3}$$

Which of the above contain (a) an Euler circuit? (b) a Hamilton circuit?
Which of the above graphs are planar?

2. *(Summer 2016)* The dodecahedron has 20 vertices and 12 faces. How many edges does the graph of the dodecahedron have?

3. *(Fall 2016)* For each of the following collection of properties, draw one graph G that satisfies them all:

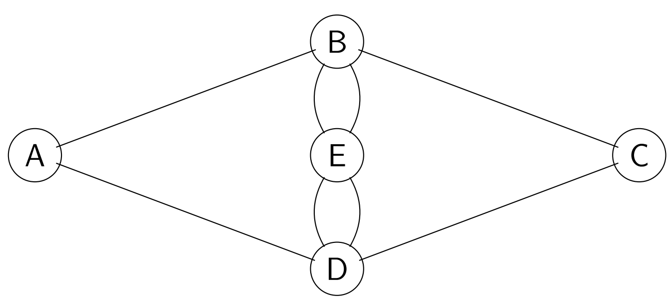
   (a) G is bipartite and contains a vertex of degree 3.

   (b) G is a non-planar graph with $\Delta(G) \leq 3$.

   (c) G is a tree with 5 vertices and $\Delta(G) = 4$.

   (d) **BONUS:** G is a non-planar graph with $\delta(G) = 1$.

4. *(Fall 2016)* Consider the following graph:



   (a) Is this graph simple? Explain why or why not.

   (b) What is the degree set for this graph? Write the degrees in increasing order.

   (c) Find an example of each of the following in the above graph or explain why they do not exist:

      i. Euler circuit

      ii. Semi-Eulerian trail

5. *(Fall 2016)*

   (a) Let G be a graph, $k \in \mathbb{Z}^+$, $k \geq 2$. Prove that if $\delta(G) = k$, then G contains a cycle of at least $k + 1$

   (b) Use the fact in part (a) to prove that every tree contains a vertex of degree 1.

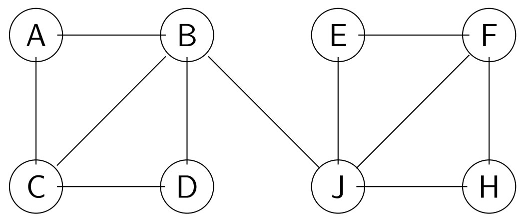
6. *(Winter 2017)* Draw each of the following graphs:

   (a) $K_{2,5}$

   (b) G where $V(G) = \{1, 2, 3, 4, 5\}$ and $E(G) = \{\{x, y\} : x \equiv y \pmod 3\}$.

   (c) G where $V(G) = \mathcal{P}(\{1, 2\})$ and $E(G) = \{\{A, B\} : A \cap B = \emptyset\}$

7. Consider the following graph G:



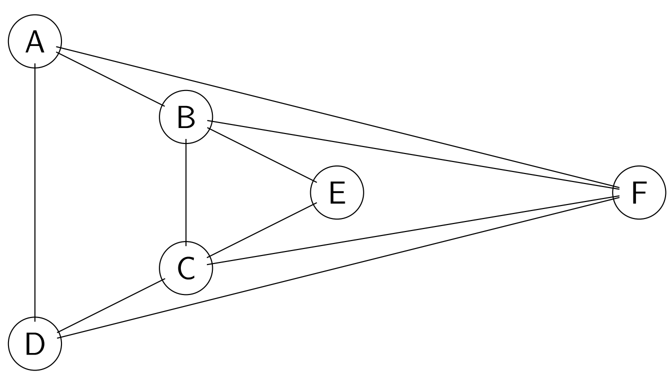
   (a) Find an open trail in G starting from A that is not a path.

   (b) Find a closed walk in G starting from A that is not a trail.

   (c) Is G planar? Justify your answer.

   (d) How many paths are there from A to H in G? Justify your answer.

8. *(Winter 2017)* Consider the following graph G:



   (a) Does G have an Euler circuit (that is, an Eulerian trail)? If so, find it. If not, justify why not.

   (b) Does G have a Hamilton cycle? If so, find it. If not, justify why not.

9. A **tree** is a connected graph which contains no cycles. A **forest** is a graph in which each component is a tree. Draw all non-isomorphic forests with at most 3 vertices. How many are there in total?

10. Let G be a simple graph with n vertices and e edges. Prove that $2e \leq n^2 - n$.

Graph Theory Solutions

1. * K_9 is the complete graph on 9 vertices.

   (a) All vertices have even degree (8), so there is an Euler circuit.

   (b) There is a Hamilton circuit.

 * K_{12} is the complete graph on 12 vertices.

   (a) All vertices have odd degree, so no Euler circuit.

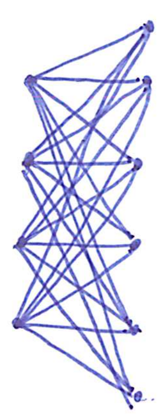
   (b) There is a Hamilton circuit.

 * C_{2016} is a cycle on 2016 vertices

   (a) All vertices have degree 2, so there is an Euler circuit.

   (b) There is a H.C.

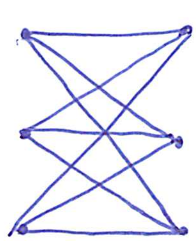
 * $K_{4,6}$:



   (a) Not all vertices have even degree $\Rightarrow$ No E.C.

   (b) No H.C.

 * $K_{3,3}$



   (a) All vertices have odd degree $\Rightarrow$ No H.C.

   (b) There is a H.C.

2. Using the formula

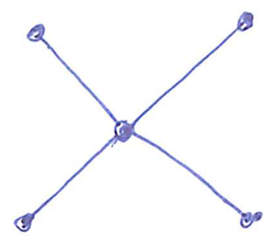
$$v - e + r = 2$$

where $v = 20$, $r = 12$

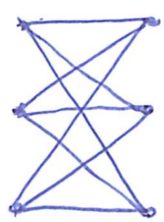
$20 - e + 12 = 2 \quad \Rightarrow \quad e = 30$

3. (a)

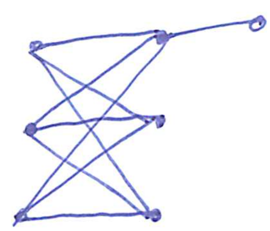
(c)



(b)



(d)



4 (a). It is not simple because there are two edges betwen B and E.

(b) $\{2, 2, 4, 4, 4\}$

(c) (i) A – B – C – D – E – B – E – D – A

(ii) There is none, because for a semi-Eulerian trail there must be exactly two vertices of odd degree

5 (a) Consider the longest path in G:

$$v_1, v_2, \ldots, v_n.$$

Consider the vertex v_1.

If v_1 has any neighbours not in the above path, then $v_1, \ldots, v_n$ is not the longest path, as that neighbour could be added to the path.

Therefore, v_1 can not have any neighbours that are not in the path.

$$\delta(G) = k \Rightarrow \deg(v_1) \geq k.$$

Let v_i be the neighbour of v_1 with largest index. Then $i \geq k+1$, so the cycle $v_1 \to v_2 \to \ldots \to v_i \to v_1$ is a cycle of length at least $k+1$.

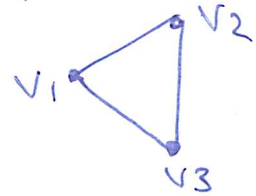
[A picture may help here:

$\underset{v_1}{\bullet} \to \underset{v_2}{\bullet} \to \ldots \to v_i \to \ldots \to v_n$

Since $\deg(v_1) \geq k$, $\Rightarrow i \geq k+1$]

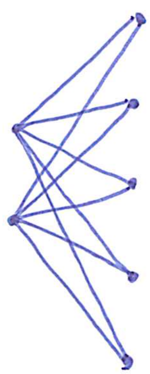
(b) Suppose there is a tree T such that $\delta(T) = 2$.
Then by part (a) there must be a cycle of length at least 3.
But a tree cannot have a cycle, so T can't be a tree

[For example, if we have

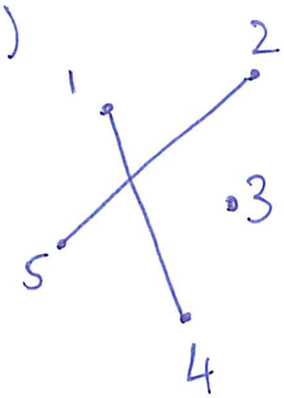


there are two paths from v_1 to v_3, which is not allowed]

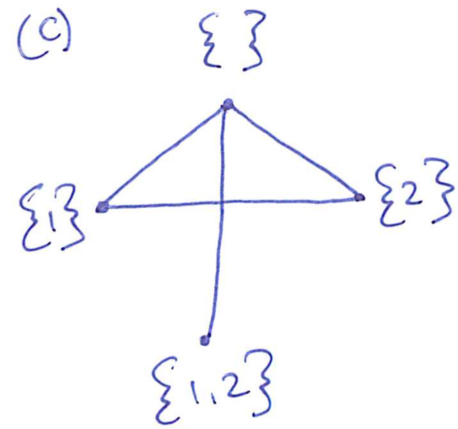
6 (a)



(b)



(c)



7. (a) $A \to B \to C \to D \to B$. (This is a trail because no edge is repeated, but it is not a path because a vertex (B) is repeated)

(b) $A \to B \to C \to D \to B \to A$ (Not a trail because the edge between A and B is repeated)

(c) It is planar because no two edges cross.

(d) To get from left side of graph to right, we need to use edge from $B \to J$.

Starting from A, to get to B, we can go from:

$$\left.\begin{array}{l} A \to B \\ A \to C \to B \\ A \to C \to D \to B \end{array}\right\} 3 \text{ ways}$$

Likewise from J to H, we can do:

$$\left.\begin{array}{l} J \to H \\ J \to F \to H \\ J \to E \to F \to H \end{array}\right\} 3 \text{ ways.}$$

So in total, $3 \times 3 = 9$ ways, so 9 paths from $A \to H$

8 (a) Question worded awkwardly:

- There is no Euler circuit because for E.C., all vertices must have even degree
- There is an Euler path because there are two vertices of odd degree:

$A \to B \to F \to C \to E \to B \to C \to D \to F \to A \to D$

(b) Hamilton cycle : $A \to B \to E \to C \to F \to D \to A$

9. <u>3 vertices</u>

1. 

2. 

3. 

<u>2 vertices</u> :

1. 

2. 

<u>1 vertex</u> :

1. 

<u>6 in total</u>.

10. $2e$ is the sum of the numbers degree set.

Let N be the sum of the numbers in the degree set.

The biggest N can be for a simple graph with n vertices is when G is complete.

Then, each vertex has degree $n-1$

$$\therefore \quad 2e = N \le (n-1) + (n-1) + \cdots + (n-1)$$
$$= n(n-1) = n^2 - n.$$